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Finite-to-one mappings and large transfinite dimension

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Abstract

Pol (1996) and Arenas (1996) independently introduced transfinite extensions of finite order of mappings by the use of the length of a partially ordered set and Borst's order, respectively. By use of the transfinite order of mappings, Arenas introduced a transfinite dimension O-dim based on the Morita's theorem and proved that every countable-dimensional compact metric space has O-dim. Then he asked whether the converse is true. In the present note, we shall show that both the transfinite extensions given by Pol and Arenas are the same if we ignore the values, and give an affirmative answer to Arenas' question as follows: a metrizable space X has the order dimension O-dim X if and only if X has large transfinite dimension Ind X. Furthermore, we shall prove that if a metrizable space X has the order dimension O-dim, then $\operatorname{Ind} X \leqslant \text{O-dim}\, X$ and $\text{O-dim}\, S_\alpha = \alpha$ for every ordinal number $\alpha < \omega_1$, where S_α is Smirnov's compactum. © 1998 Elsevier Science B.V.

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1. Introduction

In [8], Morita proved a fundamental theorem on the dimension and closed mappings in metrizable spaces: *for a metrizable space X,* $\dim X \leqslant n$ *if and only if there are a metrizable space Z with* $\dim Z = 0$ *and a closed mapping f of Z onto X such that every fiber of f contains at most $n+1$ points.* The theorem has many applications to infinite-dimensional spaces. In particular, the theorem is extended to countable-dimensional spaces, strongly

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countable-dimensional spaces and spaces that have large transfinite dimension by Nagata [9], Engelking [4] and Hattori [6], respectively.

Quite recently, Pol [13] introduced a transfinite extension of the order of finite-to-one mappings and investigated the behavior of weakly infinite-dimensional compacta under a continuous mapping with the transfinite order. Independently, Arenas [1] gave another transfinite extension of the order of finite-to-one mappings by the use of Borst's order. Then, he extended the covering dimension to transfinite dimension O-dim based on Morita's theorem and proved that every countable-dimensional compact metric space has O-dim. He asked if every compact metric space having O-dim is countable-dimensional [1, Question 3.9]. In Section 2, we show that both the transfinite extensions given by Pol and Arenas are the same if we ignore the values and they are closely related to have large transfinite dimension. Furthermore, we prove that a metrizable space X has the order dimension $\operatorname{O-dim} X$ if and only if X has large transfinite dimension $\operatorname{Ind} X$ (Theorem 2.7). This answers Arenas' question affirmatively. In Section 3, we investigate the values of the order dimension $\operatorname{O-dim} X$ for certain classes of metrizable spaces. We prove that if a metrizable space X has the order dimension O-dim, then

$$\operatorname{Ind} X \leqslant \operatorname{O-dim} X.$$

Furthermore, we also prove that

$$\operatorname{O-dim} S_\alpha = \alpha \quad \text{for every ordinal number } \alpha < \omega_1,$$

where S_α is Smirnov's compactum.

We refer the reader to [5,10] for the terminology and basic results on the theory of infinite-dimensional spaces.

2. Transfinite order of mappings

We begin with the description of Borst's order. To classify the weakly infinite-dimensional spaces, Borst [2] introduced a transfinite order $\operatorname{Ord} M$ of a collection M of finite subsets of a set.

Definition 2.1 [2]. Let L be a set, $\operatorname{Fin} L$ the collection of all nonempty finite subsets of L and M a subset of $\operatorname{Fin} L$. Let $\alpha > 0$ be an ordinal number. For $\sigma \in \{\emptyset\} \cup \operatorname{Fin} L$, we put

$$M^\sigma = \{\tau \in \operatorname{Fin} L\colon\ \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}.$$

If $\sigma = \{a\}$, we write $M^\sigma = M^a$. Then the order $\operatorname{Ord} M$ of M is inductively defined as follows:

(1) $\operatorname{Ord} M = 0$ if $M = \emptyset$.

(2) $\operatorname{Ord} M \leqslant \alpha$ if $\operatorname{Ord} M^a < \alpha$ for each $a \in L$.

If $\operatorname{Ord} M \leqslant \alpha$ for some ordinal number α, we say that $\operatorname{Ord} M$ exists (or M has $\operatorname{Ord} M$).

By use of this order, Arenas [1] defined an order of mappings as follows.

Definition 2.2 [1]. Let X, Y be topological spaces and $f: X \to Y$ a mapping. Let $\mathcal{T}(X)$ be the topology of X. We put

$$O(f) = \left\{ \tau = \{U_1, \ldots, U_n\} \in \operatorname{Fin} \mathcal{T}(X)\colon\ \overline{U_i} \cap \overline{U_j} = \emptyset \text{ for } i \neq j \right.$$
$$\left. \text{and } \bigcap_{i=1}^{n} f\big(\overline{U_i}\big) \neq \emptyset \right\}.$$

Then the order A-Ord f of f in the sense of Arenas is defined as A-Ord $f = \operatorname{Ord} O(f)$.

Let f be a mapping from a regular space X to a topological space Y. Arenas showed that if A-Ord f exists, then $f^{-1}(y)$ is countably compact for each $y \in Y$ [1, Lemma 2.2]. He asked whether every fiber of f is finite under the same assumption [1, Question 2.4]. The following is the affirmative answer of the question and it will be used later.

Proposition 2.3. *Let f be a mapping from a regular space X to a topological space Y. If* A-Ord f *exists, then $f^{-1}(y)$ is finite for each $y \in Y$.*

Proof. We suppose that there is $y \in Y$ such that $f^{-1}(y)$ is infinite. Since $f^{-1}(y)$ is countably compact [1, Lemma 2.2], there is an accumulation point x of $f^{-1}(y)$. It is easy to see that there are a sequence $\{x_n\}_{n=1}^{\infty}$ of points of $f^{-1}(y) \setminus \{x\}$ and a sequence $\{U_n\}_{n=1}^{\infty}$ of open sets of X such that $x_i \in U_i$ and $\overline{U_i} \cap (\bigcup_{j<i} \overline{U_j} \cup \{x\}) = \emptyset$ for each $i = 1, 2, \ldots$. For each i we put $\tau_i = \{U_1, \ldots, U_i\}$. Then, $\tau_i \in O(f)$ and $\tau_i \subset \tau_{i+1}$ for each i. It follows from [2, Lemma 2.1.3] that $\operatorname{Ord} O(f)$ does not exist. This is a contradiction. □

Now, we describe another approach to extend the order of finite-to-one mappings due to Pol [13]. Let $\prec$ be a partial order on a set A. We say that $\prec$ is *well-founded* on a subset C of A if there is no infinite descending sequence $a_1 \succ a_2 \succ \cdots$ in C. For a well-founded subset $(C, \prec)$ we inductively define a *length* of C as follows:

For each $a \in C$, $\operatorname{rank}_C a = 1$ if and only if there is no $b \in C$ with $b \prec a$, and

$$\operatorname{rank}_C a = \sup\{\operatorname{rank}_C b + 1\colon\ b \in C,\ \text{and } b \prec a\}.$$

Furthermore, we define $\operatorname{length} C = \sup\{\operatorname{rank}_C a\colon\ a \in C\}$.

Definition 2.4 [13]. Let $\mathcal{A}$ be a finite closed cover of a space X. Then, $\mathcal{A}$ is said to be a *regular partition* of X if each member of $\mathcal{A}$ is regular closed and $\{\operatorname{Int} A\colon\ A \in \mathcal{A}\}$ is pairwise disjoint. Let $\Gamma(X)$ be the set of all regular partitions of X. For each $\mathcal{A}$, $\mathcal{B} \in \Gamma(X)$, we define $\mathcal{A} \prec \mathcal{B}$ if $\mathcal{A}$ is a refinement of $\mathcal{B}$ and there are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \neq B$ and $A \subset \operatorname{Int} B$. Then $\prec$ is a partial order on $\Gamma(X)$.

Let f be a continuous mapping from a space X onto a space Y. We put

$$\Gamma(f) = \left\{ \mathcal{A} \in \Gamma(X)\colon\ \bigcap \{f(\operatorname{Int} A)\colon\ A \in \mathcal{A}\} \neq \emptyset \right\}.$$

Then, we say that f *has the P-order* (or P-Ord f *exists*) if $\prec$ is well-founded on $\Gamma(f)$.

A mapping $f : X \to Y$ is said to be *point finite* if every fiber of f is finite. We say that a mapping $f : X \to Y$ is *strongly point finite* [6] if $\sup\{|f^{-1}(y)|\colon\ y \in F\} < \infty$ for every closed discrete subset F of Y.

Then we have the following:

Theorem 2.5. *Let f be a closed continuous mapping from a normal space X to a T_1-space Y. Then the following are equivalent:*

(a) *f is strongly point finite.*

(b) A-Ord f *exists, i.e.,* $\operatorname{Ord} O(f)$ *exists.*

(c) P-Ord f *exists, i.e., $\prec$ is well-founded on $\Gamma(f)$.*

Proof. (a) $\Rightarrow$ (b) We suppose that f is strongly point finite and $\operatorname{Ord} O(f)$ does not exist. It follows from [2, Lemma 2.1.3] that there is a sequence $\{U_n\}_{n=1}^{\infty}$ of open sets of X such that $\overline{U_n} \cap \overline{U_m} = \emptyset$ if $n \neq m$ and $\bigcap_{i=1}^{n} f(\overline{U_i}) \neq \emptyset$ for every n. For each n we choose $y_n \in \bigcap_{i=1}^{n} f(\overline{U_i})$ and we put $F = \{y_n\colon\ n = 1, 2, \ldots\}$. It is obvious that $\sup\{|f^{-1}(y)|\colon\ y \in F\} = \infty$. To show that F is a closed discrete subset of Y, let $y \in Y$. Since $f^{-1}(y)$ is finite, there is an n_0 such that $y \notin f(\overline{U_{n_0}})$. Put

$$V = Y \setminus \bigl(f(\overline{U_{n_0}}) \cup \{y_n\colon\ n \leqslant n_0 \text{ and } y \neq y_n\}\bigr).$$

Then V is a neighborhood of y such that $|V \cap F| \leqslant 1$. This implies the closed discreteness of F. Hence f is not strongly point finite.

(b) $\Rightarrow$ (a) We suppose that $\operatorname{Ord} O(f)$ exists and f is not strongly point finite. Then there is a closed discrete subset F of Y such that $\sup\{|f^{-1}(y)|\colon\ y \in F\} = \infty$. For each n we choose $y_n \in F$ such that $|f^{-1}(y_n)| \geqslant n$. Since f is point finite, by Proposition 2.3, it follows that $y_n \neq y_m$ if $n \neq m$. Let $x_1^n, \ldots, x_n^n \in f^{-1}(y_n)$ such that $x_i^n \neq x_j^n$ if $i \neq j$. Then $\{x_i^n\colon\ i \leqslant n \text{ and } n = 1, 2, \ldots\}$ is a closed discrete in X. Since X is normal, for each $i \leqslant n$ and $n = 1, 2, \ldots$ there is an open set U_i^n of X such that $\overline{U_i^n} \cap \overline{U_j^m} = \emptyset$ if $(n, i) \neq (m, j)$. For each i we put $U_i = \bigcup_{n \geqslant i} U_i^n$ and $\tau_i = \{U_1, \ldots, U_i\}$. It is easy to see that $\tau_i \in O(f)$ and $\tau_i \subset \tau_{i+1}$ for each i. Hence $\operatorname{Ord} O(f)$ does not exist, by [2, Lemma 2.1.3].

(a) $\Rightarrow$ (c) We suppose that f is strongly point finite and $\prec$ is not well-founded on $\Gamma(f)$. There is an infinite descending sequence $\mathcal{A}_1 \succ \mathcal{A}_2 \succ \cdots$ in $\Gamma(f)$. It follows from [13, Lemma 4.1] that there are an increasing sequence $n(1) < n(2) < \cdots$ of natural numbers and disjoint subfamilies $\mathcal{D}_i \subset \mathcal{A}_{n(i)}$ such that each $\mathcal{D}_i$ contains at least i elements and $\bigcap\{f(A)\colon\ A \in \mathcal{D}_{i+1}\} \subset \bigcap\{f(A)\colon\ A \in \mathcal{D}_i\}$. We choose $y_i \in \bigcap\{f(A)\colon\ A \in \mathcal{D}_i\}$ for each i, and we put $F = \{y_i\colon\ i = 1, 2, \ldots\}$. It is clear that $\sup\{|f^{-1}(y)|\colon\ y \in F\} = \infty$. Let $y \in Y$. Since $|f^{-1}(y)| < \infty$, there is i_0 such that $y \notin \bigcap\{f(A)\colon\ A \in \mathcal{D}_{i_0}\}$. Then, there is $A_{i_0} \in \mathcal{D}_{i_0}$ such that $f^{-1}(y) \cap A_{i_0} = \emptyset$. Since f is closed, there is an open neighborhood U of y such that $f^{-1}(U) \cap A_{i_0} = \emptyset$. We put $V = U \setminus \{y_i\colon\ i \leqslant i_0 \text{ and } y_i \neq y\}$. For each $i \geqslant i_0$,

$$y_i \in \bigcap\{f(A)\colon\ A \in \mathcal{D}_i\} \subset \bigcap\{f(A)\colon\ A \in \mathcal{D}_{i_0}\} \subset f(A_{i_0}).$$

Therefore, $y_i \notin U \supset V$. Hence it follows that F is a closed discrete subset of Y, and hence f is not strongly point finite. This is a contradiction.

(c) $\Rightarrow$ (a) Suppose that f is not strongly point finite. Let $F = \{y_i\colon i = 1,2,\ldots\}$ be a closed discrete subset of Y such that $|f^{-1}(y_i)| \geqslant i$ for each i. For each i we choose i many distinct points $x_1^i,\ldots,x_i^i$ of $f^{-1}(y_i)$. Then $\{x_j^i\colon j = 1,2,\ldots,i$ and $i = 1,2,\ldots\}$ is a countable closed discrete subset of a normal space X. Hence for each $j \leqslant i$ and each $i = 1,2,\ldots,$ there is an open set U_j^i of X such that $x_j^i \in U_j^i$ and $\{U_j^i\colon j = 1,2,\ldots,i$ and $i = 1,2,\ldots\}$ is discrete in X. For each $j = 1,2,\ldots$ we put $U_j = \bigcup_{i=j}^{\infty} U_j^i$ and $\mathcal{A}_j = \{\overline{U_1},\ldots,\overline{U_j}, X \setminus (U_1 \cup \cdots \cup U_j)\}$. It is clear that $\mathcal{A}_j$ is a regular partition of X and $\mathcal{A}_{j+1} \prec \mathcal{A}_j$. Since $y_{j+1} \in \bigcap\{f(A)\colon A \in \mathcal{A}_j\}$, $\mathcal{A}_j \in \Gamma(f)$. Hence $\prec$ is not well-founded on $\Gamma(f)$. This completes the proof. □

We only consider the transfinite order due to Arenas in the rest of this paper and hence we use the symbol $\operatorname{Ord} f$ in the sense of A-$\operatorname{Ord} f$. By use of the transfinite order of mappings, Arenas extended the covering dimension to a transfinite dimension as follows.

Definition 2.6 [1]. Let X be a Tychonoff space and α an ordinal number. Then X has the order dimension O-$\dim X \leqslant \alpha$ if and only if there are a strongly zero-dimensional space Z and a perfect mapping f of Z onto X such that $\operatorname{Ord} f \leqslant \alpha + 1$.

We say that X *has the order dimension* O-$\dim X$ (or O-$\dim X$ *exists*) if O-$\dim X \leqslant \alpha$ for some ordinal number α.

The following is an affirmative answer of [1, Question 3.9] and gives a characterization of spaces that have large transfinite dimension in terms of finite-to-one closed continuous mappings.

Theorem 2.7. *A metrizable space X has the order dimension* O-dim *if and only if X has large transfinite dimension* Ind.

Proof. It follows from [6, Theorem 2.6] that a metrizable space X has large transfinite dimension Ind if and only if there are a metrizable space Z with $\dim Z = 0$ and a strongly point finite closed continuous mapping f of Z onto X. Hence the 'if' part is a direct consequence of Theorem 2.5. To show the 'only if' part, we suppose that X has the order dimension O-$\dim X$. Then there are a Tychonoff space Z and a perfect mapping f of Z onto X such that $\dim Z = 0$ and $\operatorname{Ord} f$ exists. It follows from Pasynkov's factorization theorem [11, Theorem 3] that there are a metrizable space Y and mappings $g\colon Z \to Y$ and $h\colon Y \to X$ such that $\dim Y = 0$ and $f = h \circ g$. Then h is a perfect mapping. To show that $\operatorname{Ord} h$ exists, let $\mathcal{T}(Y)$ and $\mathcal{T}(Z)$ denote the topologies of Y and Z respectively. We define a mapping $\Phi\colon \mathcal{T}(Y) \to \mathcal{T}(Z)$ as $\Phi(U) = g^{-1}(U)$ for each $U \in \mathcal{T}(Y)$. It is easy to see that for each $\sigma = \{U_1,\ldots,U_n\} \in O(h)$, $\Phi(\sigma) \in O(f)$ and $|\Phi(\sigma)| = |\sigma|$. Hence, by [2, Lemma 2.1.6], $\operatorname{Ord} O(h)$ exists and $\operatorname{Ord} O(h) \leqslant \operatorname{Ord} O(f)$. Hence Theorem 2.5 and [6, Theorem 2.6] imply that X has large transfinite dimension. This completes the proof. □

Remark 2.8. Arenas mentioned the following [1, Theorem 3.4]: Let α be an ordinal number. If X is a topological sum of a family of spaces $\{X_\lambda\colon \lambda \in \Lambda\}$ such that

O-dim $X_\lambda \leqslant \alpha$ for each $\lambda \in \Lambda$, then O-dim $X \leqslant \alpha$. But, this is false. Indeed, let X be the topological sum of n-cubes I^n, $n = 1, 2, \ldots$. Then it is well known that X does not have large transfinite dimension. Hence, it follows from Theorem 2.7 above that X does not have order dimension O-dim. On the other hand, it is easy to see that if a space $X = X_1 \oplus X_2$, then O-dim $X = \max(\text{O-dim}\, X_1, \text{O-dim}\, X_2)$. For, let Z be a strongly 0-dimensional space and f be a perfect mapping from Z onto X. Since X_i ($i = 1, 2$) is open and closed in X, $O(f|_{f^{-1}(X_i)}) \subset O(f)$ for $i = 1, 2$. This means that O-dim $X_i \leqslant$ O-dim X for $i = 1, 2$. Conversely, let Z_1 and Z_2 be strongy 0-dimensional spaces, f_1 a perfect mapping from Z_1 onto X_1 and f_2 a perfect mapping from Z_2 onto X_2. Put $Z = Z_1 \oplus Z_2$ and $f = f_1 \oplus f_2$. Then Z is a 0-dimensional space and f is a perfect mapping from Z onto X. Moreover it is easy to check that $O(f) \subset \widetilde{O}(f_1) \cup \widetilde{O}(f_2)$, where

$$\widetilde{O}(f_i) = \Big\{\sigma = \{U_1, U_2, \ldots, U_n\} \in \operatorname{Fin} \mathcal{T}(Z)\colon\ \overline{U_j} \cap \overline{U_k} = \emptyset \text{ for } j \neq k \\ \text{and } \bigcap_{j=1}^{n} f\big(\overline{U_j} \cap Z_i\big) \neq \emptyset\Big\}$$

for $i = 1, 2$. By [2, Lemmas 2.1.6 and 2.2.1], we have that $\operatorname{Ord} \widetilde{O}(f_i) \leqslant \operatorname{Ord} f_i$ for $i = 1, 2$ and $\operatorname{Ord} f \leqslant \max(\operatorname{Ord} f_1, \operatorname{Ord} f_2)$. Therefore these mean that

$$\text{O-dim}\, X \leqslant \max(\text{O-dim}\, X_1, \text{O-dim}\, X_2).$$

By Remark 2.8 above, the proof of Example 3.8 in [1] is not correct. Hence, the following question seems to be open.

Question 2.9. *Let X be a compact space having the order dimension* O-dim X. *Is X countable-dimensional?*

3. The values of the order dimension O-dim

We further the investigation of the relation between the order dimension and large transfinite dimension. We begin with some lemmas.

Lemma 3.1. *Let X be a metrizable space and A a closed subset of X. Let $\mathcal{T}(X)$ and $\mathcal{T}(A)$ be families of all open sets of X and A, respectively. Then there is a mapping $\Phi: \mathcal{T}(A) \to \mathcal{T}(X)$ such that $\Phi(U) \cap A = U$ for each $U \in \mathcal{T}(A)$ and $\overline{\Phi(U)} \cap \overline{\Phi(V)} = \emptyset$ for each $U, V \in \mathcal{T}(A)$ with $\overline{U} \cap \overline{V} = \emptyset$.*

Proof. The lemma may be already known. We show an outline of the proof for the convenience of the reader. For each $U \in \mathcal{T}(A)$ and each $u \in U$ there is $\varepsilon(u, U) > 0$ such that $S_{\varepsilon(u,U)}(u) \cap A \subset U$, where $S_\delta(u)$ is the δ-neighborhood of u. Then we put

$$\Phi(U) = \bigcup \big\{S_{\varepsilon(u,U)/3}(u)\colon\ u \in U\big\}.$$

It is easy to see that Φ satisfies the conditions of the lemma. $\square$

Remark 3.2. By use of Lemma 3.1, we have a subset theorem of the order dimension O-dim for metrizable spaces: if A is a closed subset of a metrizable space X having O-dim X, then A has O-dim and O-dim $A \leqslant$ O-dim X. This is a partial answer of [1, Question 3.10]. However, we do not know whether if the subset theorem of O-dim holds for nonmetrizable spaces.

We need a sum theorem for large transfinite dimension to prove Lemma 3.3 below. For each ordinal number α, we denote $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number and $n(\alpha)$ is a nonnegative integer. Let α and β be ordinal numbers. Then the lower sum $\alpha \oplus \beta$ of α and β is given as follows:

$$\alpha \oplus \beta = \begin{cases} \alpha, & \text{if } \lambda(\alpha) > \lambda(\beta), \\ \alpha + n(\beta) = \beta + n(\alpha), & \text{if } \lambda(\alpha) = \lambda(\beta), \\ \beta, & \text{if } \lambda(\alpha) < \lambda(\beta). \end{cases}$$

Lemma 3.3 [7,12,3]. *Let X be a hereditarily normal space. If there are closed subsets F_1 and F_2 of X such that $X = F_1 \cup F_2$ and F_1 and F_2 have large transfinite dimension, then*

$$\operatorname{Ind} X \leqslant \max(\operatorname{Ind} F_1, \operatorname{Ind} F_2) \oplus \big(\operatorname{Ind}(F_1 \cap F_2) + 1\big).$$

Lemma 3.4. *Let X be a metrizable space having large transfinite dimension and α a limit ordinal number less than ω_1. If $\operatorname{Ind} X \geqslant \alpha$, then there are a sequence $\{U_n\}_{n=1}^{\infty}$ of open sets of X and a sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets of X such that*

(1) $U_n \cap U_m = \emptyset$ *if* $n \neq m$,
(2) $F_n \subset U_n$ *for each* n,
(3) $\operatorname{Ind} F_n \leqslant \operatorname{Ind} F_{n+1}$, *and*
(4) $\sup_{n\to\infty} \operatorname{Ind} F_n = \alpha$.

Proof. Since $\operatorname{Ind} X \geqslant \alpha$, there is a closed subset F of X such that $\operatorname{Ind} F = \alpha$. Hence, by making use of Lemma 3.1, we can assume that $\operatorname{Ind} X = \alpha$. Let $\{\beta_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of ordinal numbers such that $\sup_{n\to\infty} \beta_n = \alpha$. Since $\operatorname{Ind} X = \alpha > \beta_n$, there are disjoint closed subsets A_n and B_n of X such that $\operatorname{Ind} C_n \geqslant \beta_n$ for every partition C_n between A_n and B_n. For each n, let $\{G_i^n\}_{i=0}^{\infty}$ be a decreasing sequence of open sets of X such that $A_n \subset G_i^n$, $\overline{G_{i+1}^n} \subset G_i^n$ for each $i \geqslant 0$ and $\overline{G_0^n} \cap B_n = \emptyset$. For each $i \geqslant 1$, there are open sets U_i^n, V_i^n and W_i^n of X such that

$$\operatorname{Bd} G_i^n \subset W_i^n \subset \overline{W_i^n} \subset V_i^n \subset \overline{V_i^n} \subset U_i^n \subset \overline{U_i^n} \subset G_{i-1}^n \setminus \overline{G_{i+1}^n}$$

and $\overline{U_i^n} \cap \overline{U_j^n} = \emptyset$ if $i \neq j$. If there is n_0 such that $\operatorname{Ind} \overline{V_i^{n_0}} = \alpha$ for each $i \geqslant 1$, then we put $F_i = \overline{V_i^{n_0}}$ and $U_i = U_i^{n_0}$ for each i. Then the sequences $\{U_i\}_{i=1}^{\infty}$ and $\{F_i\}_{i=1}^{\infty}$ are desired.

We suppose that for each n there is $i(n) \geqslant 1$ such that $\operatorname{Ind} \overline{V_{i(n)}^n} < \alpha$. By the induction, for each natural number p we shall choose natural numbers $\ell(p)$ and $k(p)$ which satisfy the following.

(5) $\ell(p) \leqslant k(p) < \ell(p+1)$ for each p, and

(6) $\operatorname{Ind} F_p \geqslant \beta_{\ell(p)}$, where $F_p = \operatorname{Bd} G^{k(p)}_{i(k(p))} \setminus \bigcup_{q=1}^{p-1} V^{k(q)}_{i(k(q))}$.
We put $\ell(1) = k(1) = 1$. Since $F_1 = \operatorname{Bd} G^1_{i(1)}$ separates A_1 and B_1 in X, it follows that $\operatorname{Ind} F_1 \geqslant \beta_1$. Let r be a natural number. We assume that there are $\ell(1), \ell(2), \ldots, \ell(r)$ and $k(1), k(2), \ldots, k(r)$ which satisfy conditions (5) and (6) above. Since

$$\operatorname{Ind} \overline{V^{k(s)}_{i(k(s))}} < \alpha \quad \text{for each } s \leqslant r,$$

it follows from Lemma 3.3 that

$$\operatorname{Ind} \bigcup_{s=1}^{r} \overline{V^{k(s)}_{i(k(s))}} < \alpha.$$

Hence there are natural numbers $\ell(r+1)$ and $k(r+1)$ such that

$$\beta_{\ell(r+1)} > \max\left(\beta_{k(r)}, \operatorname{Ind} \bigcup_{s=1}^{r} \overline{V^{k(s)}_{i(k(s))}}\right), \quad \text{and}$$

$$\beta_{k(r+1)} > \beta_{\ell(r+1)} + n\left(\operatorname{Ind} \bigcup_{s=1}^{r} \overline{V^{k(s)}_{i(k(s))}}\right).$$

We suppose that $\operatorname{Ind} F_{r+1} < \beta_{\ell(r+1)}$. Then, by Lemma 3.3, it follows that

$$\begin{aligned}
\operatorname{Ind} \operatorname{Bd} G^{k(r+1)}_{i(k(r+1))} &\leqslant \operatorname{Ind}\left(F_{r+1} \cup \bigcup_{s=1}^{r} \overline{V^{k(s)}_{i(k(s))}}\right) \\
&\leqslant \operatorname{Ind} F_{(r+1)} \oplus \left(\operatorname{Ind} \bigcup_{s=1}^{r} \overline{V^{k(s)}_{i(k(s))}} + 1\right) \\
&\leqslant \beta_{\ell(r+1)} + n\left(\operatorname{Ind} \bigcup_{s=1}^{r} \overline{V^{k(s)}_{i(k(s))}}\right) < \beta_{k(r+1)}.
\end{aligned}$$

This contradicts the choice of $A_{k(r+1)}$ and $B_{k(r+1)}$. Hence, it follows that

$$\operatorname{Ind} F_{r+1} \geqslant \beta_{\ell(r+1)} > \operatorname{Ind} \overline{V^{k(r)}_{i(k(r))}} \geqslant \operatorname{Ind}\left(\operatorname{Bd} G^{k(r)}_{i(k(r))}\right) \geqslant \operatorname{Ind} F_r.$$

We put

$$U_{r+1} = W^{k(r+1)}_{i(k(r+1))} \setminus \bigcup_{s=1}^{r} \overline{W^{k(s)}_{i(k(s))}}.$$

Then $F_{r+1} \subset U_{r+1}$ and $U_s \cap U_{r+1} = \emptyset$ for each $s \leqslant r$. Hence the inductive procedure is complete and so the lemma is proved. $\square$

Theorem 3.5. *Let X be a metrizable space having the order dimension* O-dim X. *Then the inequality* $\operatorname{Ind} X \leqslant$ O-dim X *holds.*

Proof. We shall prove the theorem by the induction on O-dim. If α be an ordinal number with $\alpha < \omega_0$ and X a metrizable space with O-dim $X \leqslant \alpha$, then $\operatorname{Ind} X =$ O-dim $X \leqslant \alpha$ by Morita's theorem. Let α be a transfinite ordinal number. We suppose that for every metrizable space Y with O-dim $Y \leqslant \beta$ for some ordinal number $\beta < \alpha$, $\operatorname{Ind} Y \leqslant \beta$ holds. Let X be a metrizable space such that O-dim $X = \alpha$. As is shown in the proof of Theorem 2.7, there are a metrizable space Y and a perfect mapping $h : Y \to X$ such that $\dim Y = 0$ and $\operatorname{Ord} h \leqslant \alpha + 1$. Let A and B be disjoint closed sets of X. Since $\dim Y = 0$, there is a clopen set G of Y such that $h^{-1}(A) \subset G$ and $G \cap h^{-1}(B) = \emptyset$. We put $H = X \setminus h(Y \setminus G)$. Then H is an open set of X such that $A \subset H$ and $\overline{H} \cap B = \emptyset$. Let $C = \operatorname{Bd} H$, $D = h^{-1}(C) \cap G$ and $g = h|_D : D \to C$. Since C is a partition between A and B, it suffices to show that $\operatorname{Ind} C < \alpha$. It is clear that $\operatorname{Ind} D = 0$ and g is a closed continuous mapping from D onto C. Let $\mathcal{T}(D)$ and $\mathcal{T}(Y)$ be the topologies of D and Y respectively. By Lemma 3.1, there is a mapping $\Phi : \mathcal{T}(D) \to \mathcal{T}(Y)$ such that $\Phi(U) \cap D = U$ for each $U \in \mathcal{T}(D)$ and $\overline{\Phi(U)} \cap \overline{\Phi(V)} = \emptyset$ for each $U, V \in \mathcal{T}(D)$ with $\overline{U} \cap \overline{V} = \emptyset$.

For each $U \in \mathcal{T}(D)$, we put $\Psi(U) = \Phi(U) \cap G$. We shall show that

$$\Psi(\sigma) = \{\Psi(U_1), \ldots, \Psi(U_n)\} \in O(h)^{\{Y \setminus G\}}$$

for each $\sigma = \{U_1, \ldots, U_n\} \in O(g)$. Indeed, it is clear that $\overline{\Psi(U_i)} \cap \overline{\Psi(U_j)} = \emptyset$ for $i \neq j$ and $\overline{\Psi(U_i)} \cap (X - G) = \emptyset$ for each i. Since $\sigma \in O(g)$, there is $x \in C$ such that $x \in \bigcap_{i=1}^n g(\overline{U_i})$. Since $x \in C$, $h^{-1}(x) \cap (Y \setminus G) \neq \emptyset$. Therefore, it follows that

$$x \in \bigcap_{i=1}^n g(\overline{U_i}) \cap h(Y \setminus G) \subset \bigcap_{i=1}^n h(\overline{\Psi(U_i)}) \cap h(Y \setminus G),$$

and hence $\Psi(\sigma) \in O(h)^{\{Y \setminus G\}}$. It is also clear that $|\Psi(\sigma)| = |\sigma|$ for each $\sigma \in O(g)$. Hence, it follows from [2, Lemma 2.1.6] that $\operatorname{Ord} O(g) \leqslant \operatorname{Ord} O(h)^{\{Y - G\}} \leqslant \alpha$.

Case 1. If $\alpha = \beta + 1$, then $\operatorname{Ord} g \leqslant \alpha = \beta + 1$. Hence O-dim $C \leqslant \beta$. Therefore, it follows that $\operatorname{Ind} C \leqslant \beta < \alpha$ by the inductive hypothesis.

Case 2. Let α be a limit ordinal number. Since C has O-dim, by Theorem 2.7, C has also large transfinite dimension $\operatorname{Ind} C$. We suppose that $\operatorname{Ind} C \geqslant \alpha$. By Lemma 3.4, there are a sequence $\{U_n\}_{n=1}^\infty$ of open sets of C and a sequence $\{F_n\}_{n=1}^\infty$ of closed sets of C such that

(1) $U_n \cap U_m = \emptyset$ if $n \neq m$,
(2) $F_n \subset U_n$ for each n,
(3) $\operatorname{Ind} F_n \leqslant \operatorname{Ind} F_{n+1}$, and
(4) $\sup_{n \to \infty} \operatorname{Ind} F_n = \alpha$.

Without loss of generality, we can assume that there is an increasing sequence $\{\beta_n\}_{n=1}^\infty$ of ordinal numbers such that $\beta_n + 1 \leqslant \operatorname{Ind} F_n$ for each n and $\sup_{n \to \infty} \beta_n = \alpha$. For each n, we put $E_n = g^{-1}(F_n)$ and $g_n = g|_{E_n} : E_n \to F_n$. Then $\operatorname{Ind} E_n = 0$ and g_n is a closed continuous onto mapping. Since $\operatorname{Ind} F_n \geqslant \beta_n + 1$, it follows that $\operatorname{Ord} g_n \geqslant \beta_n + 2$ by the inductive assumption. Hence there is an open set W_n of E_n such that $\operatorname{Ord} O(g_n)^{W_n} \geqslant \beta_n + 1$. Let V_n be an open set of C such that $F_n \subset V_n \subset \overline{V_n} \subset U_n$. By Lemma 3.1, there is a mapping $\Phi_n : \mathcal{T}(E_n) \to \mathcal{T}(D)$ such that $\Phi_n(W) \cap E_n = W$ for each $W \in \mathcal{T}(E_n)$ and $\overline{\Phi_n(W)} \cap \overline{\Phi_n(W')} = \emptyset$ for each $W, W' \in \mathcal{T}(E_n)$ with $\overline{W} \cap \overline{W'} = \emptyset$. For each

$W \in \mathcal{T}(E_n)$, we put $\Psi_n(W) = \Phi_n(W) \cap g^{-1}(V_n)$ and $\widetilde{W} = \bigcup_{n=1}^{\infty} \Psi_n(W_n)$. Then it is easy to see that for each n and each $\sigma = \{W_n^1, \ldots, W_n^k\} \in O(g_n)^{W_n}$ it follows that

$$\Psi_n(\sigma) = \{\Psi_n(W_n^1), \ldots, \Psi_n(W_n^k)\} \in O(g)^{\widetilde{W}}$$

and $|\Psi_n(\sigma)| = |\sigma|$. Hence it follows from [2, Lemma 2.1.6] that

$$\beta_n + 1 \leqslant \operatorname{Ord} O(g_n)^{W_n} \leqslant \operatorname{Ord} O(g)^{\widetilde{W}}.$$

Therefore,

$$\operatorname{Ord} O(g)^{\widetilde{W}} \geqslant \sup_{n \to \infty} (\beta_n + 1) = \alpha.$$

Hence $\operatorname{Ord} O(g) \geqslant \alpha + 1$. This is a contradiction. Hence $\operatorname{Ind} C < \alpha$. This completes the proof. □

Concerning the result above, it is natural to ask the following:

Question 3.6. *Let X be a metrizable space having large transfinite dimension* $\operatorname{Ind} X$. *Does the inequality* O-dim $X \leqslant \operatorname{Ind} X$ *hold*?

We conclude the paper by proving O-dim $S_\alpha = \alpha$, where S_α is the Smirnov's compactum. Smirnov's compacta $S_0, S_1, \ldots, S_\alpha, \ldots$, $\alpha < \omega_1$, are defined by transfinite induction: $S_0 = I^0 = \{0\}$ is a one-point space, and, for $\alpha > 0$,

$$S_\alpha = \begin{cases} S_\beta \times I & \text{if } \alpha = \beta + 1, \\ \bigoplus_{\beta < \alpha} S_\beta \cup \{c_\alpha\} & \text{if } \alpha \text{ is a limit ordinal,} \end{cases}$$

where $\bigoplus_{\beta<\alpha} S_\beta \cup \{s_\alpha\}$ is the one-point compactification of $\bigoplus_{\beta<\alpha} S_\beta$.

For each $n \in \mathbb{N} \cup \{0\}$, since $\dim I^n = n$, let C_n be a metrizable space and g_n a closed mapping from C_n onto I^n such that $\dim C_n = 0$ and every fiber of g_n contains at most $n+1$ points. Then we have the following.

Lemma 3.7. *For each ordinal number $\alpha < \omega_1$ and finite sequence $n_1, n_2, \ldots, n_s$ of nonnegative integers there are a metrizable space Y_α and a perfect mapping $h_\alpha : Y_\alpha \to S_{\lambda(\alpha)}$ such that* $\dim Y_\alpha = 0$ *and*

$$\operatorname{Ord}\left(h_\alpha \times g_{n(\alpha)} \times \prod_{j=1}^{s} g_{n_j}\right) < \lambda(\alpha) + \omega_0.$$

Proof. We prove the lemma by the induction of $\alpha < \omega_1$. For each finite ordinal and each finite sequence of nonnegative integers the lemma is clear. So, let α be an infinite countable ordinal number. We suppose that for each ordinal number less than α and each finite sequence of nonnegative integers the lemma is true. Fix a finite sequence of nonnegative integers $n_1, n_2, \ldots, n_s$. Since $S_{\lambda(\alpha)} = \bigoplus_{\beta<\lambda(\alpha)} S_\beta \cup \{c_{\lambda(\alpha)}\}$, from our inductive assumption, for each $\beta < \lambda(\alpha)$ and the sequence $n(\alpha), n_1, \ldots, n_s$, there are a metrizable space Y_β and a perfect mapping $h_\beta : Y_\beta \to S_{\lambda(\beta)}$ such that $\dim Y_\beta = 0$ and

$$\operatorname{Ord}\left(h_\beta \times g_{n(\beta)} \times g_{n(\alpha)} \times \prod_{j=1}^{s} g_{n_j}\right) < \lambda(\beta) + \omega_0. \tag{1}$$

Put $Z_\beta = Y_\beta \times C_{n(\beta)}$ and $f_\beta = h_\beta \times g_{n(\beta)}$ for each $\beta < \lambda(\alpha)$,

$$Y_\alpha = \bigoplus_{\beta<\lambda(\alpha)} Z_\beta \cup \{z_{\lambda(\alpha)}\}$$

and $h_\alpha : Y_\alpha \to S_{\lambda(\alpha)}$ be a mapping defined by $h_\alpha|_{Z_\beta} = f_\beta$ for each $\beta < \lambda(\alpha)$ and $h_\alpha(z_{\lambda(\alpha)}) = c_{\lambda(\alpha)}$. Then Y_α is a metrizable space with $\dim Y_\alpha = 0$ and h_α is a perfect mapping.

We shall show that

$$\operatorname{Ord}\left(h_\alpha \times g_{n(\alpha)} \times \prod_{j=1}^{s} g_{n_j}\right) < \lambda(\alpha) + \omega_0.$$

For convenience, we put

$$p = \big(n(\alpha)+1\big) \times \prod_{j=1}^{s}(n_j+1), \qquad q = n(\alpha) + \sum_{j=1}^{s} n_j,$$

$$C = C_{n(\alpha)} \times \prod_{j=1}^{s} C_{n_j}, \qquad g = g_{n(\alpha)} \times \prod_{j=1}^{s} g_{n_j}.$$

Take an arbitrary

$$\sigma = \{U_1, U_2, \ldots, U_{p+1}\} \in O(h_\alpha \times g).$$

Then the family $\{\overline{U_1}, \overline{U_2}, \ldots, \overline{U_{p+1}}\}$ is pairwise disjoint and $\bigcap_{i=1}^{p+1} h_\alpha \times g(\overline{U_i}) \neq \emptyset$. Let $A = \bigcap_{i=1}^{p+1} h_\alpha \times g(\overline{U_i})$. Since we can consider that

$$Y_\alpha \times C_{n(\alpha)} \times \prod_{j=1}^{s} C_{n_j} = Y_\alpha \times C = \bigoplus_{\beta<\lambda(\alpha)} Z_\beta \times C \cup \{z_{\lambda(\alpha)}\} \times C \quad \text{and}$$

$$S_\alpha \times \prod_{j=1}^{s} I^{n_j} = S_{\lambda(\alpha)} \times I^q = \bigoplus_{\beta<\lambda(\alpha)} S_\beta \times I^q \cup \{c_{\lambda(\alpha)}\} \times I^q,$$

$h_\alpha \times g|_{\{z_{\lambda(\alpha)}\}\times C}$ is a p-to-1 mapping, and hence it follows that $A \cap (\{c_{\lambda(\alpha)}\} \times I^q) = \emptyset$. Since A is compact there are finite $\beta_1, \beta_2, \ldots, \beta_m < \lambda(\alpha)$ with $\beta_1 < \beta_2 < \cdots < \beta_m$ such that

$$A \subset \bigoplus_{t=1}^{m} S_{\beta_t} \times I^q \quad \text{and} \tag{2}$$

$$A \cap (S_{\beta_t} \times I^q) \neq \emptyset \quad \text{for } t = 1, 2, \ldots, m. \tag{3}$$

For each $t = 1, 2, \ldots, m$, we put

$$\widetilde{O}(f_{\beta_t} \times g) = \left\{ \tau = \{W_1, W_2, \ldots, W_l\} \in \operatorname{Fin} \mathcal{T}(Y_\alpha \times C)\colon \ \{\overline{W_1}, \overline{W_2}, \ldots, \overline{W_l}\} \text{ is pairwise disjoint, and } \bigcap_{i=1}^{l} f_{\beta_t} \times g\big(\overline{W_i} \cap (Z_{\beta_t} \times C)\big) \neq \emptyset \right\}.$$

From property (3), $\sigma \in \widetilde{O}(f_{\beta_t} \times g)$ for each $t = 1, 2, \ldots, m$. Moreover we can see that $O(h_\alpha \times g)^\sigma \subset \bigcup_{t=1}^{m} \widetilde{O}(f_{\beta_t} \times g)^\sigma$. For, let $\tau = \{W_1, W_2, \ldots, W_l\} \in O(h_\alpha \times g)^\sigma$. Then $\tau \cap \sigma = \emptyset$, $\{\overline{W_1}, \overline{W_2}, \ldots, \overline{W_l}\} \cup \{\overline{U_1}, \overline{U_2}, \ldots, \overline{U_{p+1}}\}$ is pairwise disjoint and

$$\bigcap_{i=1}^{l} h_\alpha \times g\big(\overline{W_i}\big) \cap \bigcap_{i=1}^{p+1} h_\alpha \times g\big(\overline{U_i}\big) \neq \emptyset.$$

By property (2),

$$\bigcap_{i=1}^{l} h_\alpha \times g\big(\overline{W_i}\big) \cap A \subset \bigoplus_{t=1}^{m} S_{\beta_t} \times I^q.$$

Hence there is $t = 1, 2, \ldots, m$ such that

$$\begin{aligned}
&\bigcap_{i=1}^{l} f_{\beta_t} \times g\big(\overline{W_i} \cap (Z_{\beta_t} \times C)\big) \cap \bigcap_{i=1}^{p+1} f_{\beta_t} \times g\big(\overline{U_i} \cap (Z_{\beta_t} \times C)\big) \\
&\quad = \bigcap_{i=1}^{l} \big(h_\alpha \times g(\overline{W_i}) \cap (S_{\beta_t} \times I^q)\big) \cap \bigcap_{i=1}^{p+1} \big(h_\alpha \times g(\overline{U_i}) \cap (S_{\beta_t} \times I^q)\big) \neq \emptyset.
\end{aligned}$$

Thus this follows that $\tau \in \widetilde{O}(f_{\beta_t} \times g)^\sigma$.

By [2, Lemma 2.2.1], we have that

$$\operatorname{Ord} O(h_\alpha \times g)^\sigma \leqslant \max\big\{\operatorname{Ord} \widetilde{O}(f_{\beta_t} \times g)^\sigma\colon \ t = 1, 2, \ldots, m\big\}.$$

Let $\Phi(U) = U \cap (Z_{\beta_t} \times C)$ for each open subset U of $Y_\alpha \times C$. Then it is easy to check that for $t = 1, 2, \ldots, m$ and $\tau = \{W_1, W_2, \ldots, W_l\} \in \widetilde{O}(f_{\beta_t} \times g)$,

$$\Phi(\tau) = \big\{\Phi(W_1), \Phi(W_2), \ldots, \Phi(W_l)\big\} \in O(f_{\beta_t} \times g)$$

and $|\tau| = |\Phi(\tau)|$. So, by [2, Lemma 2.1.6], $\operatorname{Ord} \widetilde{O}(f_{\beta_t} \times g) \leqslant \operatorname{Ord} O(f_{\beta_t} \times g)$. Since

$$f_{\beta_t} = h_{\beta_t} \times g_{n(\beta)} \quad \text{and} \quad g = g_{n(\alpha)} \times \prod_{j=1}^{s} g_{n_j},$$

from property (1),

$$\begin{aligned}
\operatorname{Ord} \widetilde{O}(f_{\beta_t} \times g)^\sigma &\leqslant \operatorname{Ord} O(f_{\beta_t} \times g)^{\Phi(\sigma)} \\
&= \operatorname{Ord} O\Big(h_{\beta_t} \times g_{n(\beta)} \times g_{n(\alpha)} \times \prod_{j=1}^{s} g_{n_j}\Big)^{\Phi(\sigma)} \\
&< \lambda(\beta_t) + \omega_0 \leqslant \lambda(\alpha),
\end{aligned}$$

for each $t = 1, 2, \ldots, m$. Therefore, we have that

$$\operatorname{Ord}\left(h_{\beta_t} \times g_{n(\beta)} \times g_{n(\alpha)} \times \prod_{j=1}^{s} g_{n_j}\right) \leqslant \lambda(\alpha) + p < \lambda(\alpha) + \omega_0.$$

This completes the proof. □

Theorem 3.8. *For each ordinal number $\alpha < \omega_1$ O-dim $S_\alpha = \alpha$.*

Proof. Since

$$\text{O-dim}\, S_\alpha \geqslant \operatorname{Ind} S_\alpha = \alpha$$

by Theorem 3.5, it suffices to show that O-dim $S_\alpha \leqslant \alpha$. For finite ordinal number n, O-dim $I^n = \operatorname{Ind} I^n = n$ and so let α be an infinite ordinal number less than ω_1. For each $\beta < \lambda(\alpha)$ and $n(\alpha) \in \mathbb{N} \cup \{0\}$, by Lemma 3.7, there are a metrizable space Y_β and a perfect mapping $h_\beta : Y_\beta \to S_{\lambda(\beta)}$ such that $\dim Y_\beta = 0$ and

$$\operatorname{Ord}(h_\beta \times g_{n(\beta)} \times g_{n(\alpha)}) < \lambda(\beta) + \omega_0. \tag{4}$$

Put $Z_\beta = Y_\beta \times C_{n(\beta)}$ and $f_\beta = h_\beta \times g_{n(\beta)}$ for each $\beta < \lambda(\alpha)$, $Z_\alpha = Y_\alpha \times C_{n(\alpha)}$ and $f_\alpha = h_\alpha \times g_{n(\alpha)} : Z_\alpha \to S_\alpha$, where $Y_\alpha = \bigoplus_{\beta<\lambda(\alpha)} Z_\beta \cup \{z_{\lambda(\alpha)}\}$ is the one-point compactification of $\bigoplus_{\beta<\lambda(\alpha)} Z_\beta$ and $h_\alpha : Y_\alpha \to S_{\lambda(\alpha)}$ is the mapping defined by $h_\alpha|_{Z_\beta} = f_\beta$ for $\beta < \lambda(\alpha)$ and $h_\alpha(z_{\lambda(\alpha)}) = c_{\lambda(\alpha)}$. Then it is easily seen that Z_α is a metrizable space with $\dim Z_\alpha = 0$ and f_α is a perfect mapping. To prove that

$$\operatorname{Ord} f_\alpha \leqslant \alpha + 1 = \lambda(\alpha) + n(\alpha) + 1,$$

take an arbitrary $\sigma = \{U_1, U_2, \ldots, U_{n(\alpha)+2}\} \in O(f_\alpha)$. Then $\{\overline{U_1}, \overline{U_2}, \ldots, \overline{U_{n(\alpha)+2}}\}$ is pairwise disjoint and $\bigcap_{i=1}^{n(\alpha)+2} f_\alpha(\overline{U_i}) \neq \emptyset$. If we put $A = \bigcap_{i=1}^{n(\alpha)+2} f_\alpha(\overline{U_i})$, with the same argument of the proof of Lemma 3.7, there is an ordinal number $\beta < \lambda(\alpha)$ such that

$$\operatorname{Ord} O(f_\alpha)^\sigma \leqslant \operatorname{Ord} O(f_\beta \times g_{n(\alpha)})^{\sigma|_{Z_\beta \times C_{n(\alpha)}}},$$

where

$$\sigma|_{Z_\beta \times C_{n(\alpha)}} = \{U_1 \cap (Z_\beta \times C_{n(\alpha)}), U_2 \cap (Z_\beta \times C_{n(\alpha)}), \ldots, U_{n(\alpha)+2} \cap (Z_\beta \times C_{n(\alpha)})\}.$$

Since $f_\beta = g_\beta \times g_{n(\beta)}$, from property (4) we have that

$$\begin{aligned}\operatorname{Ord} O(f_\beta \times g_{n(\alpha)})^{\sigma|_{Z_\beta \times C_{n(\alpha)}}} &= \operatorname{Ord} O(h_\beta \times g_{n(\beta)} \times g_{n(\alpha)})^{\sigma|_{Z_\beta \times C_{n(\alpha)}}} \\ &< \lambda(\beta) + \omega_0 \leqslant \lambda(\alpha).\end{aligned}$$

Consequently, we have that $\operatorname{Ord} f_\alpha \leqslant \lambda(\alpha) + n(\alpha) + 1 = \alpha + 1$, and hence the theorem is proved. □

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